Convex Counterparts to Iterative Transform Algorithms and New Methods for Phase Retrieval

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In collaboration with H. H. Bauschke and P. L. Combettes
Phase Retrieval

Given
Phase Retrieval

Find
Phase Retrieval: wavefront reconstruction for NGST

Given
Phase Retrieval: wavefront reconstruction for NGST

Find
Phase Retrieval

Initial guess
Phase Retrieval: algorithms
Phase Retrieval: algorithms

- Iterative Transform: Misell (averaged projections) (‘73), Error Reduction (ER), Basic Input-Output (BIO), Hybrid Input-Output (HIO) (Fienup, ‘76, ‘82),
- Transport of intensity equations and PDE’s with nonstandard data (Teague, ‘73; Gureyev, Roberts & Nugent, ‘95, Gureyev & Nugent ‘96)
- Gradient Descent: Nonlinear, “nonsmooth” extended least squares (L., Burke, Lyon, ‘02)
- Hybrid Projection-Reflection (HPR) (Bauschke, Combettes, L., 2003)
- Relaxed Average Successive Reflections (RASR) (Bauschke, Combettes, L., 2003)
Phase Retrieval Algorithms: convergence analysis
Phase Retrieval Algorithms: convergence analysis

- ER is monotone decreasing. (Fienup, 1982)

- Gerchberg-Saxton (ER) is a nonconvex sequential projection algorithm (Levi& Stark, ’84, ’87)

- Under certain conditions, a smoothly perturbed set distance error admits a locally convergent descent algorithm whose steps are arbitrarily close to averaged projections (L., Burke, Lyon, 2002)

- For linear object domain constraints, Fienup’s BIO and HIO algorithms correspond to nonconvex instances of the Dykstra and Douglas-Rachford algorithms respectively (Bauschke, Combettes, L., 2002)

- Challenges:
Phase Retrieval Algorithms: convergence analysis

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- Under certain conditions, a smoothly perturbed set distance error admits a locally convergent descent algorithm whose steps are arbitrarily close to averaged projections (L., Burke, Lyon, 2002)

- For linear object domain constraints, Fienup’s BIO and HIO algorithms correspond to nonconvex instances of the Dykstra and Douglas-Rachford algorithms respectively (Bauschke, Combettes, L., 2002)

- Challenges: NONCONVEXITY and nonlinearity
Phase Retrieval Algorithms: convergence analysis

Our Approach:

Convex analysis ↔ algorithms ↔ nonconvex applications
Iterative Transform Algorithms: HIO, HPR and RASR

Preliminaries

Original signal: \( x_\ast \in \mathcal{H} : \mathbb{R}^2 \rightarrow \mathbb{R} \)

Estimated signal: \( x \in \mathcal{H} : \mathbb{R}^2 \rightarrow \mathbb{R} \)
Preliminaries

Data: \( m \in \mathcal{H} : \mathbb{R}^2 \to \mathbb{R}_+ \)

Support of \( x_* \subset D \subset \mathbb{R}^2 \)
Preliminaries

Constraint sets:
Preliminaries

Constraint sets:

- Generic sets: $A, B, \text{ and } C \subset H$, *a priori* information (closed convex)
Preliminaries

Constraint sets:

- Generic sets: $A, B, \text{ and } C \subset \mathcal{H}, \ a \text{ proiri information }$ (closed convex)
- Support constraint: $S = \{ y \in \mathcal{H} \mid y \cdot 1_{\mathcal{L}_D} = 0 \}$ (convex)
- Support and nonegativity: $S_+ = \{ y \in \mathcal{H} \mid y \cdot 1_{\mathcal{L}_D} = 0 \ \text{and} \ y \geq 0 \}$ (convex)
Preliminaries

Constraint sets:

- Generic sets: $A, B, \text{ and } C \subset \mathcal{H}$, \textit{a priori} information (closed convex)

- Support constraint: $S = \{ y \in \mathcal{H} \mid y \cdot 1_{\mathcal{C}_D} = 0 \}$ (convex)

- Support and nonegativity: $S_+ = \{ y \in \mathcal{H} \mid y \cdot 1_{\mathcal{C}_D} = 0 \text{ and } y \geq 0 \}$ (convex)

- (Fourier) Magnitude: $M = \{ y \in \mathcal{H} \mid |\mathcal{F}y| = m \}$ (nonconvex)
Preliminaries

- (Fourier) Magnitude:
Convex Analysis: projectors, reflectors and feasibility problems

- Projectors: $P_C : \mathcal{H} \to \mathcal{H}$, for any $x \in \mathcal{H}$, $P_C x = c \in C$ is the nearest member of $C$ to $x$. 
Convex Analysis: projectors, reflectors and feasibility problems

- **Projectors:** $P_C : \mathcal{H} \rightarrow \mathcal{H}$, for any $x \in \mathcal{H}$, $P_C x = c \in C$ is the nearest member of $C$ to $x$.

- **Reflectors:** $R_C = 2P_C - I$
Convex Analysis: projectors, reflectors and feasibility problems

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Convex Analysis: projectors, reflectors and feasibility problems

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- Feasibility problem: Given the sets $A, B$, find $x \in A \cap B$
Convex Analysis: projectors, reflectors and feasibility problems

- Projectors: \( P_C : \mathcal{H} \rightarrow \mathcal{H} \), for any \( x \in \mathcal{H} \), \( P_C x = c \in C \) is the nearest member of \( C \) to \( x \).

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- Fix point sets: for any operator \( T \) on \( \mathcal{H} \), \( \text{Fix} \, T = \{ x \in \mathcal{H} \mid T(x) = x \} \)

- Feasibility problem: Given the sets \( A, B \), find \( x \in A \cap B \)

Note: \( A \cap B \) might be empty (inconsistent feasibility problem)
Nonconvex Application: phase retrieval

- Support projector (linear and convex): $P_S(x) = x \cdot 1_D$
Nonconvex Application: phase retrieval

- Support projector (linear and convex): $P_S(x) = x \cdot 1_D$

- Nonnegativity and support projector (nonlinear and convex):

$$\forall t \in \mathbb{R}^2 \quad (P_{S+}(x))(t) = \begin{cases} \max\{0, x(t)\}, & \text{if } t \in D; \\ 0, & \text{otherwise} \end{cases}$$
Nonconvex Application: phase retrieval

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- Nonnegativity and support projector (nonlinear and convex):
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  \max\{0, x(t)\}, & \text{if } t \in D; \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Fourier magnitude projector (nonlinear and nonconvex):
  \[
  P_M(x) = \mathcal{F}^{-1}(\hat{y}_0), \quad \text{where} \quad \hat{y}_0(\omega) = \begin{cases} 
  m(\omega)\frac{\mathcal{F}x(\omega)}{|\mathcal{F}x(\omega)|}, & \text{if } \mathcal{F}x(\omega) \neq 0; \\
  m(\omega), & \text{otherwise}
  \end{cases}
  \]
Nonconvex Application: phase retrieval

- Nonconvex feasibility problem with two sets:
Nonconvex Application: phase retrieval

- Nonconvex feasibility problem with two sets:

  (i) find $x \in S \cap M$. 
Nonconvex Application: phase retrieval

- **Nonconvex** feasibility problem with two sets:

  (i) **find** $x \in S \cap M$

  (ii) **find** $x \in S_+ \cap M$
Convex Analysis: POCS

- Alternating Projections Onto Convex Sets (POCS) (Bregman, 1965): for any $x_0 \in \mathcal{H}$,

$$x_{n+1}(t) = (P_A P_B x_n)(t)$$
Convex Analysis: POCS

- **Alternating Projections Onto Convex Sets (POCS) (Bregman, 1965):** for any $x_0 \in \mathcal{H}$,
  $$x_{n+1}(t) = (P_A P_B x_n)(t)$$

- **Fact (Cheney&Goldstein, 1959, Bregman, 1965, Gurin,Polyak,& Raik,1967):** For $A \cap B \neq \emptyset$, both sequences $a_n$ and $b_n$ converge weakly to a point in $A \cap B$. Under certain conditions (e.g. compactness), convergence is strong.
Nonconvex Extensions: ER and PONCS

• Alternating Projections Onto Nonconvex Sets (PONCS) (Levi & Stark 1984). If we replace $A$ and $B$ with $S_+$ and $M$ respectively, we get the Error Reduction algorithm. For all $t \in \mathbb{R}^2$

$$x_{n+1}(t) = (P_{S_+} P_M x_n)(t) = \begin{cases} (P_M(x_n))(t), & \text{if } t \in D \text{ and } P_M(x_n)(t) > 0; \\
0, & \text{otherwise} \end{cases}$$
Nonconvex Extensions: gradient descent and PONCS

- Averaged projections: for any $x_0 \in \mathcal{H}$ and $\gamma_1 + \gamma_2 = 1$

$$x_{n+1} = (\gamma_1 P_A + \gamma_2 P_B) (x_n)$$
Nonconvex Extensions: gradient descent and PONCS

- Averaged projections: for any $x_0 \in \mathcal{H}$ and $\gamma_1 + \gamma_2 = 1$

  $$x_{n+1} = (\gamma_1 P_A + \gamma_2 P_B) (x_n)$$

- **Fact:** In the convex setting, the average of the projections is the gradient of the sum of square set distances, i.e. the above is a gradient descent algorithm. This fact was heuristically applied to the nonconvex setting of phase retrieval (Fienup, 1982, Barakat& Newsam 1985), however, the nonconvex projection is multivalued and belongs to the subgradient of the sum of square set distances (L., 2001).
Convex Analysis: Dykstra’s algorithm

- Dykstra’s algorithm (1983): for $A$ a closed vector space, fix a starting point $x_0$, set $q_{-1} = 0$

\[
b_n = P_B(x_n + q_{n-1}), \quad q_n = (I - P_B)(x_n + q_{n-1}), \quad x_{n+1} = P_A(b_n)
\]

hence

\[
x_{n+1} + q_n = (P_A P_B + I - P_B)(x_n + q_{n-1}) = (P_A P_B + I - P_B)^{n+1}(x_0)
\]
Convex Analysis: Dykstra’s algorithm

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\]

- Fact (Boyle-Dykstra, 1985): For $A \cap B \neq \emptyset$, both $x_n$ and $b_n$ converge in norm to $P_{A \cap B}(x_0)$, the point in $A \cap B$ closest to $x_0$. 
Nonconvex Extensions: BIO and Nonconvex Dykstra

- **Fact** (Bauschke, Combettes, L. 2002): Replacing $A$ with $S$ and $B$ with $M$, Dykstra’s algorithm coincides with Fienup’s BIO algorithm: for all $t \in \mathbb{R}^2$

\[
x_{n+1}(t) = \begin{cases} 
    x_n(t), & \text{if } t \in D \\
    x_n(t) - (P_B(x_n))(t), & \text{otherwise}
\end{cases}
\]

if and only if

\[
x_{n+1} + q_n = (P_S P_M + I - P_M)^{n+1}(x_0) \quad \text{where} \quad q_n = (I - P_M)(x_n + q_{n-1})
\]
Convex Analysis: ASR

- The Douglas-Rachford/Lions-Mercier/Averaged Successive Reflection (ASR) algorithm (Douglas&Rachford 1956, and Lions&Mercier, 1979, Bauschke, Combettes,& L., 2003): for $A$ a closed vector space, given any $x_0 \in H$

$$x_{n+1} = \frac{1}{2}(R_AR_B + I)(x_n)$$
Iterative Transform Algorithms: HIO, HPR and RASR

Convex Analysis: ASR

- The Douglas-Rachford/Lions-Mercier/Averaged Successive Reflection (ASR) algorithm (Douglas&Rachford 1956, and Lions&Mercier, 1979, Bauschke, Combettes,& L., 2003): for $A$ a closed vector space, given any $x_0 \in \mathcal{H}$

$$x_{n+1} = \frac{1}{2}(R_AR_B + I)(x_n)$$

- Fact (Lions& Mercier, 1979): For $A \cap B \neq \emptyset$, the iterates $x_n$ converge weakly to a point $x \in \text{Fix}\left(\frac{1}{2}(R_AR_B + I)\right)$ and $P_B(x) \in A \cap B$. Moreover, the sequence $(P_B(x_n))$ is bounded, and every weak cluster point of $(P_B(x_n))$ lies in $A \cap B$. If $\mathcal{H}$ is finite-dimensional, then $x_n \to x$ and $P_B(x_n) \to P_B(x) \in A \cap B$. 
Convex Analysis: ASR

• What if $A \cap B = \emptyset$?
Convex Analysis: ASR

- What if $A \cap B = \emptyset$? Need some tools. Let

$$D = \overline{B - A}, \quad v = P_D(0), \quad E = A \cap (B - v), \quad \text{and} \quad F = (A + v) \cap B.$$
Convex Analysis: ASR

- What if $A \cap B = \emptyset$? Need some tools. Let

$$D = \overline{B - A}, \quad v = P_D(0), \quad E = A \cap (B - v), \quad \text{and} \quad F = (A + v) \cap B.$$ 

(i) $\|v\| = \inf \|A - B\|$ (i.e. $v$ is the “gap” vector), and the infimum is attained if and only if $v \in B - A$

(ii) $E = \text{Fix}(P_A P_B)$ and $F = \text{Fix}(P_B P_A)$, $E + v = F$

(iii) If $e \in E$ and $f \in F$, then $P_B e = P_F e = e + v$ and $P_A f = P_E f = f - v$

(iv) $E$ and $F$ are nonempty provided one of the following conditions holds: (a) $A \cap B \neq \emptyset$; (b) $B - A$ is closed; (c) $A$ or $B$ is bounded; (d) $A$ and $B$ are polyhedral sets;

In other words: $v$ is the gap vector, $E$ is the set of points in $A$ nearest to $B$, and $F$ is the set of points in $B$ nearest to $A$. 
Convex Analysis: ASR

- **Fact** (Bauschke, Combettes, & L. 2003): If $A \cap B = \emptyset$ then

  $$\text{Fix } \left( \frac{1}{2} (R_AR_B + I) \right) = \emptyset.$$ 

  In any case, $\text{Fix } \left( \frac{1}{2} (R_AR_B + I) + v \right)$ is closed, convex, and

  $$F + N_D(v) \subset \text{Fix } \left( \frac{1}{2} (R_AR_B + I) + v \right) \subset v + F + N_D(v).$$

  In particular, if $A \cap B = \emptyset$, then $\text{Fix } \left( \frac{1}{2} (R_AR_B + I) + v \right)$ is fuzzy.
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  In particular, if $A \cap B = \emptyset$, then $\text{Fix} \left( \frac{1}{2} (R_AR_B + I) + v \right)$ is **fuzzy**.

  Hence, for $x_{n+1} = \frac{1}{2} (R_AR_B + I)(x_n)$

  (i) $x_n - x_{n+1} \to v$ and $P_Bx_n - P_AP_Bx_n \to v$

  (ii) If $A \cap B = \emptyset$, then $\|x_n\| \to +\infty$.  


Convex Analysis: HPR

- Hybrid Projection-Reflection (HPR)  
  (H. Bauschke, P. Combettes, & R. L., 2002): given any $x_0 \in \mathcal{H}$

$$x_{n+1} = \frac{1}{2}(R_A(R_B + (\beta - 1)P_B) + I + (1 - \beta)P_B)(x_n)$$
Convex Analysis: HPR

- **Hybrid Projection-Reflection (HPR)**  
  (H. Bauschke, P. Combettes, & R. L., 2002): given any $x_0 \in \mathcal{H}$

  $$x_{n+1} = \frac{1}{2} \left( R_A (R_B + (\beta - 1) P_B) + I + (1 - \beta) P_B \right)(x_n)$$

- **Fact:** For $\beta = 1$ HPR and ASR coincide.
Nonconvex Extensions: HPR and HIO

Replacing $A$ with $S$ and $B$ with $M$, HPR coincides with Fienup’s HIO algorithm:

$$
x_{n+1} = \frac{1}{2} \left( R_S (R_M + (\beta - 1)P_M) + I + (1 - \beta)P_M \right) (x_n)
$$

if an only if for all $t \in \mathbb{R}^2$

$$
x_{n+1}(t) = \begin{cases} 
(P_M(x_n))(t), & \text{if } t \in D \\
x_n(t) - \beta (P_M(x_n))(t), & \text{otherwise}
\end{cases}
$$
Nonconvex Extensions: HPR and HIO

Replacing \(A\) with \(S_+\) and \(B\) with \(M\), HPR does not coincide with Fienup’s HIO algorithm! In this case, HPR yields, for all \(t \in \mathbb{R}^2\),

\[
x_{n+1}(t) = \begin{cases} 
(P_M(x_n))(t), & \text{if } t \in D \text{ and } (R_M(x_n))(t) \geq (1 - \beta)(P_M(x_n))(t) \\
 x_n(t) - \beta(P_M(x_n))(t), & \text{otherwise}
\end{cases}
\]

HIO, on the other hand, is given by

\[
x_{n+1}(t) = \begin{cases} 
(P_M(x_n))(t), & \text{if } t \in D \text{ and } (P_M(x_n))(t) \geq 0 \\
 x_n(t) - \beta(P_M(x_n))(t), & \text{otherwise}
\end{cases}
\]
Convex Analysis: RASR

- **Relaxed Averaged Successive Reflections (RASR)** (Bauschke, Combettes, & L., 2003): Let

\[ T_{\beta_n} = \beta_n \frac{1}{2} (R_A R_B + I) + (1 - \beta_n) P_B, \quad 0 < \beta < 1. \]

Then, given any \( x_0 \in \mathcal{H} \)

\[ x_{n+1} = T_{\beta_n} x_n. \]
Convex Analysis: RASR
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- **Fact**: For $\beta = 1$, RASR, ASR and HPR coincide and the fixed point set is fuzzy if $A \cap B = \emptyset$. 
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● **Fact:** For $0 < \beta < 1$, the fixed point set of $T_\beta$ in RASR is sharp (even if $A \cap B = \emptyset$): $\text{Fix } T_\beta = F - \frac{\beta}{1-\beta} v,$
Convex Analysis: RASR

- **Fact**: For $\beta = 1$, RASR, ASR and HPR coincide and the fixed point set is fuzzy if $A \cap B = \emptyset$.

- **Fact**: For $0 < \beta < 1$, the fixed point set of $T_{\beta}$ in RASR is sharp (even if $A \cap B = \emptyset$): $\text{Fix } T_{\beta} = F - \frac{\beta}{1-\beta}v$.

- **Fact**: For $0 < \beta < 1$, the iterates converge weakly for any reasonable approximate evaluation of the operator, and for $x_n = T_{\beta}^n x_0$,

  (i) $x_n - x_{n+1} \to 0$ and $P_B x_n - P_A P_B x_n \to v$

  (ii) If $F = \emptyset$, then $\|x_n\|$ is unbounded.
Convex Analysis: RASR

- **Fact:** For $\beta = 1$, RASR, ASR and HPR coincide and the fixed point set is fuzzy if $A \cap B = \emptyset$.

- **Fact:** For $0 < \beta < 1$, the fixed point set of $T_\beta$ in RASR is sharp (even if $A \cap B = \emptyset$): $\text{Fix} T_\beta = F - \frac{\beta}{1-\beta} v$.

- **Fact:** For $0 < \beta < 1$, the iterates converge weakly for any reasonable approximate evaluation of the operator, and for $x_n = T_\beta^n x_0$,

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  (ii) If $F = \emptyset$, then $\|x_n\|$ is unbounded.

- (For HPR with $\beta \neq 1$, nothing is known.)
Nonconvex extensions: RASR and phase retrieval

Replacing $A$ with $S_+$ and $B$ with $M$, yields the following algorithm

$$x_{n+1} = \left( \frac{\beta_n}{2}(R_{S+}R_M + I) + (1 - \beta_n)P_M \right)(x_n)$$

if and only if, for all $t \in \mathbb{R}^2$,

$$x_{n+1}(t) = \begin{cases} 
(P_M(x_n))(t), & \text{if } t \in D \text{ and } (R_M(x_n))(t) \geq 0 \\
\beta_n x_n(t) - (1 - 2\beta_n)(P_M(x_n))(t), & \text{otherwise}
\end{cases}$$
HIO, HPR, RASR: practical implementation

- **RMS error:**

\[
E_{\text{RMS}}(x_n) = \frac{\| \mathcal{F}x_n - m \|}{\| m \|},
\]

In convex case, if \( A \cap B = \emptyset \) then \( E_{\text{RMS}} \) is unbounded

- **Gap distance:**

\[
E_{S+}(x_n) = \frac{\| P_{S_+}(P_M(x_n)) - P_M(x_n) \|^2}{\| m \|^2}.
\]

In convex case, \( E_A \) will always converge to the gap distance between \( A \) and \( B \).
HIO, HPR, RASR: a comparison

- Asymptotic behavior, no noise

$\beta = 1.0$
HIO, HPR, RASR: a comparison

- Asymptotic behavior, no noise

\[ \beta = 1.0 \]
HIO, HPR, RASR: a comparison

- Asymptotic behavior, SNR 34dB

\[ \beta = 0.75 \quad \beta_n = 0.75 \rightarrow 1.0 \]
ITERATIVE TRANSFORM ALGORITHMS: HIO, HPR AND RASR

HIO, HPR, RASR: a comparison

- Asymptotic behavior, SNR $34\text{dB}$

\[
\beta = 0.75 \quad \text{and} \quad \beta_n = 0.75 \rightarrow 1.0
\]
HIO, HPR, RASR: a comparison

- Asymptotic behavior, SNR 34dB

\[ \beta = .75 \]

\[ \beta_n = 0.75 \rightarrow 1.0 \]
HIO, HPR, RASR: a comparison

• Typical image, $\beta = 0.75$, SNR 34dB, 40 iterations

<table>
<thead>
<tr>
<th>HIO</th>
<th>HPR</th>
<th>RASR</th>
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HIO, HPR, RASR: a comparison

- Typical image, $\beta = 0.75$, SNR $34\text{dB}$, 40 iterations

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![Image with HIO, HPR, RASR results]
HIO, HPR, RASR: a comparison

- Typical image, $\beta = 0.75$, SNR 34dB, 40 iterations
HIO, HPR, RASR: a comparison

- Typical image, $\beta = 0.75$, SNR 34dB, 40 iterations
HIO, HPR, RASR: a comparison

- Typical images, $\beta_n = 0.75 \rightarrow 1.0$, SNR 34dB, 40 iterations
HIO, HPR, RASR: a comparison

- Typical images, $\beta_n = 0.75 \rightarrow 1.0$, SNR 34dB, 40 iterations
HIO, HPR, RASR: a comparison

- Typical images, $\beta_n = 0.75 \rightarrow 1.0$, SNR 34dB, 40 iterations
HIO, HPR, RASR: a comparison

- Typical images, $\beta_n = 0.75 \rightarrow 1.0$, SNR 34dB, 40 iterations
HIO, HPR=RASR=ASR: a comparison

- Typical images, $\beta_n = 1.0$, SNR $34$ dB, 200 iterations